The Schurman Parabola - Part II

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In Part I we developed the mathematics to model a revenue stream or asset base that grows at an unsustainable short-term growth rate that decreases to a sustainable long-term growth rate over time. In Part II we will combine our parabolic growth rate model with an exponentiall constant growth rate model to value a company.

Assume that we have a revenue stream where the short-term growth rate is unstainably high and decreases to a sustainable long-term growth rate over time. In this white paper we will develop a closed-form valuation equation that can handle a revenue stream or asset base with a non-constant growth rate. To develop the mathematics we will use the following hypothetical problem...

Table 1 - Hypothetical Problem

Description	Parameter	Value
Annualized revenue at time zero	R_0	\$1,000,000
Short-term annual growth rate	μ_S	0.50
Long-term annual growth rate	μ_L	0.04
Transition period (in years)	T	7.00
Revenue contribution margin	θ	0.20
Assets as a percent of annualized revenue	ϕ	0.60
Income tax rate	α	0.30
Discount rate	κ	0.15

In the problem above we have a company where revenue growth declines from an unsustainable short-term growth rate to a sustainable long-term growth rate over time. The goal of this white paper will be to develop the mathematics to answer the following question...

Question: What is the DCF (discounted cash flow) value of this company?

Revenue

Annualized revenue at time $t \le T$ grows at the unsustainable short-term revenue growth rate μ_S which decreases to the sustainable revenue growth rate μ_L over the time interval $[0, T]$. Using the equation for a parabola, which we developed in Part I, and the parameters in Table 1 the equation for annualized revenue at time $t \le T$ is...

$$R_t = a\,t^2 + b\,t + c \text{ ...when... } 0 \le t \le T \tag{1}$$

Note that the equations for parabola parameters from Part I are...

$$a = \frac{\mu_L\,(bT + c) - b}{2T - \mu_L\,T^2} \text{ ...and... } b = \mu_S\,c \text{ ...and... } c = R_0 \tag{2}$$

Annualized revenue at time $t > T$ grows at the sustainable growth rate μ_L. Using the parameters in Table 1 the equation for annualized revenue at time $t > T$ is...

$$R_t = R_T \operatorname{Exp}\left\{\mu_L \left(t - T\right)\right\} \text{ ...when... } t > T \tag{3}$$

Investment

We will define total assets to be a function of annualized revenue. Using the parameters in Table 1 above the equation for total assets at time $t \geq 0$ is...

$$A_t = \phi\, R_t \tag{4}$$

Given that our annualized revenue equation for time $t \leq T$ is different than our annualized revenue equation for time $t > T$ we need two equations for both total assets and the change in total assets. Using annualized revenue Equations (1) and (4) above and the parameters in Table 1 the equation for total assets at time $t \leq T$ is...

$$A_t = \phi\, R_t = \phi\bigg(a\,t^2 + b\,t + c\bigg) \;\text{...when...}\; 0 \leq t \leq T \tag{5}$$

The equation for the change in total assets is the first derivative of the total asses equation with respect to time. Using Equation (5) above the equation for the change in total assets at time $t \leq T$ is...

$$\frac{\delta A_t}{\delta t} = \phi\bigg(2\,a\,t + b\bigg) \;\text{...when...}\; 0 \leq t \leq T \tag{6}$$

Note that we can rewrite Equation (6) as...

$$\delta A_t = \phi\bigg(2\,a\,t + b\bigg)\delta t \;\text{...when...}\; 0 \leq t \leq T \tag{7}$$

Using annualized revenue Equations (3) and (4) above and the parameters in Table 1 the equation for total assets at time $t > T$ is...

$$A_t = \phi\, R_t = \phi\Bigg(R_T\,\mathrm{Exp}\bigg\{\mu_L\,\bigg(t - T\bigg)\bigg\}\Bigg) \;\text{...when...}\; t > T \tag{8}$$

Using Equation (8) above the equation for the change in total assets at time $t > T$ is...

$$\frac{\delta A_t}{\delta t} = \phi\,\mu_L\Bigg(R_T\,\mathrm{Exp}\bigg\{\mu_L\,\bigg(t - T\bigg)\bigg\}\Bigg) \;\text{...when...}\; t > T \tag{9}$$

Note that we can rewrite Equation (9) as...

$$\delta A_t = \phi\,\mu_L\Bigg(R_T\,\mathrm{Exp}\bigg\{\mu_L\,\bigg(t - T\bigg)\bigg\}\Bigg)\delta t \;\text{...when...}\; t > T \tag{10}$$

Net Income

We will define the variable N_t to be net income over the time interval $[t, t + \delta t]$. Using the parameters in Table 1 the equation for net income is...

$$N_t = \theta\,\bigg(1 - \alpha\bigg)\,R_t\,\delta t \tag{11}$$

Given that our annualized revenue equation for time $t \leq T$ is different than our annualized revenue equation for time $t > T$ we need two equations for net income. Using annualized revenue Equations (1) and the parameters in Table 1 the equation for net income at time $t \leq T$ is...

$$N_t = \theta\,\bigg(1 - \alpha\bigg)\bigg(a\,t^2 + b\,t + c\bigg)\delta t \;\text{...when...}\; 0 \leq t \leq T \tag{12}$$

Using annualized revenue Equations (3) and the parameters in Table 1 the equation for net income at time $t > T$ is...

$$N_t = \theta\,\bigg(1 - \alpha\bigg)R_T\,\mathrm{Exp}\bigg\{\mu_L\,\bigg(t - T\bigg)\bigg\}\,\delta t \;\text{...when...}\; t > T \tag{13}$$

Net Cash Flow

We will define the variable C_t to be net cash flow over the time interval $[t, t+\delta t]$. Using Equations (4) and (11) above the equation for net cash flow is...

$$C_t = N_t - \delta A_t \tag{14}$$

Given that our annualized revenue equation for time $t \le T$ is different than our annualized revenue equation for time $t > T$ we need two equations for net cash flow. Using Equations (7) and (12) above the equation for net cash flow at time $t \le T$ is...

$$C_t = \theta \Big(1-\alpha\Big)\Big(a\,t^2 + b\,t + c\Big)\delta t - \phi\Big(2\,a\,t + b\Big)\delta t \;\text{ ...when... }\; 0 \le t \le T \tag{15}$$

Using Equations (10) and (13) above the equation for net cash flow at time $t > T$ is...

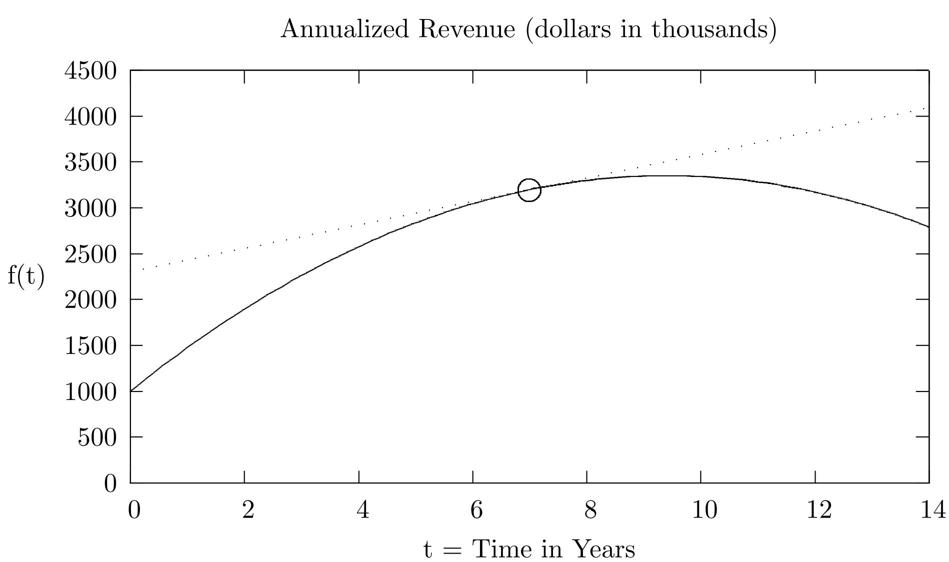
$$C_t = \theta\Big(1-\alpha\Big) R_T \,\mathrm{Exp}\Big\{\mu_L\Big(t-T\Big)\Big\}\delta t - \phi\,\mu_L\Big(R_T\,\mathrm{Exp}\Big\{\mu_L\Big(t-T\Big)\Big\}\Big)\delta t \;\text{ ...when... }\; t > T \tag{16}$$

Valuation

We will define the variable V to be the present value of cash flows over the time interval $[0, \infty]$. We will define the variable $V_{0,T}$ to be the present value of cash flows over the time interval $[0, T]$ and the variable $V_{T,\infty}$ to be the present value of cash flows over the time interval $[T, \infty]$. The equation for company value is...

$$V = V_{0,T} + V_{T,\infty} \tag{17}$$

We will use our parabolic model to value cash flow received over the time interval $[0, T]$ and an exponential constant growth rate model to value cash flow received over the time interval $[T, \infty]$. Note that in the graphic below we will use the parabola (solid line) to value cash flows received prior to time T and the tangent (dashed line) to value cash flows received subsequent to time T...



Using Equation (15) above the equation for the present value of cash flow received over the period in which the revenue growth rate transitions from the unsustainable short-term growth rate to the sustainable long-term growth rate is...

$$
\begin{aligned}
V_{0,T} &= \int\limits_0^T C_t \,\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\
&= \int\limits_0^T \theta\left(1-\alpha\right)\left(a\,t^2+b\,t+c\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t - \int\limits_0^T \phi\left(2\,a\,t+b\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\
&= \theta\left(1-\alpha\right)\int\limits_0^T \left(a\,t^2+b\,t+c\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t - \phi\int\limits_0^T \left(2\,a\,t+b\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t
\end{aligned} \tag{18}
$$

We will define I_1 to be the solution to the first integral in Equation (18) above. Using Appendix Equation (32) the solution to the integral is...

$$
\begin{aligned}
I_1 = \int\limits_0^T \left(a\,t^2+b\,t+c\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(a\left(\kappa^2 T^2+2\kappa T+2\right)\kappa^{-3}+b\left(\kappa T+1\right)\kappa^{-2}+c\kappa^{-1}\right) \\
+\,2a\kappa^{-3}+b\kappa^{-2}+c\kappa^{-1}
\end{aligned} \tag{19}
$$

We will define I_2 to be the solution to the second integral in Equation (18) above. Using Appendix Equation (33) the solution to the integral is...

$$
I_2 = \int\limits_0^T \left(2\,a\,t+b\right)\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(2a\left(\kappa T+1\right)\kappa^{-2}+b\,\kappa^{-1}\right)+2a\kappa^{-2}+b\kappa^{-1} \tag{20}
$$

Using Equation (16) above the equation for the present value of cash flow received over the period in which the revenue growth rate is constant is..

$$
\begin{aligned}
V_{T,\infty} &= \int\limits_T^\infty C_t \,\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\
&= \int\limits_T^\infty \theta\left(1-\alpha\right)R_T\,\mathrm{Exp}\left\{\mu_L\left(t-T\right)\right\}\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t - \int\limits_T^\infty \phi\,\mu_L R_T\,\mathrm{Exp}\left\{\mu_L\left(t-T\right)\right\}\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\
&= \left(\theta\left(1-\alpha\right)-\phi\,\mu_L\right)R_T\int\limits_T^\infty \mathrm{Exp}\left\{\mu_L\left(t-T\right)-\kappa\,t\right\}\delta t
\end{aligned} \tag{21}
$$

We will define I_3 to be the solution to the integral in Equation (21) above. Using Appendix Equation (34) the solution to the integral is...

$$
I_3 = \int\limits_T^\infty \mathrm{Exp}\left\{\mu_L\left(t-T\right)-\kappa\,t\right\}\delta t = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(\mu_L-\kappa\right)^{-1} \tag{22}
$$

Using Equations (18) through (22) above we can rewrite our company value Equation (17) as...

$$
V = \theta\left(1-\alpha\right)I_1 - \phi\,I_2 + R_T\left(\theta\left(1-\alpha\right)-\phi\,\mu_L\right)I_3 \tag{23}
$$

The Answer To Our Hypothetical Problem

Our first step will be to solve for parabola parameter c. Using parabola equations from Part I and the data in Table 1 above the value of parabola parameter c is...

$$
c = R_0 = 1000000 \tag{24}
$$

Our next step will be to solve for parabola parameter b. Using parabola equations from Part I, Equation (24) above and the data in Table 1 the value of parabola parameter b is...

$$b = \mu_S\, c = (0.50)(1000000) = 500000 \tag{25}$$

Our next step will be to solve for parabola parameter c. Using parabola equations from Part I, Equations (24) and (25) above and the data in Table 1 the value of parabola parameter a is...

$$a = \frac{\mu_L\,(bT + c) - b}{2T - \mu_L\, T^2} = \frac{(0.04)((500000)(7) + 1000000) - 500000}{(2)(7) - (0.04)(7)^2} = -26578 \tag{26}$$

When we transition from the non-constant growth rate model to the constant growth rate model at time T we will need annualized revenue at time T. Using Equations (24), (25) and (26) above and the data in Table 1 the equation for annualized revenue at time T is...

$$R_T = at^2 + bt + c = (-26578)(7)^2 + (500000)(7) + 1000000 = 3197674 \tag{27}$$

Using Equation (19) and the parameters above the solution to integral one is...

$$\begin{aligned} I_1 = &-\text{Exp}\,\{(-0.15)(7)\}((-26578)((0.15)^2(7)^2 + (2)(0.15)(7) + 2)(0.15)^{-3} + (500000)((0.15)(7) + 1)(0.15)^{-2} \\ &+ (1000000)(0.15)^{-1}) + (2)(0.15)(7)) + (2)(-26578)(0.15)^{-3} + (500000)(0.15)^{-2} + (1000000)(0.15)^{-1} \\ &= 9201209 \end{aligned} \tag{28}$$

Using Equation (20) and the parameters above the solution to integral two is...

$$\begin{aligned} I_2 = &-\text{Exp}\,\{(-0.15)(7)\}((2)(-26578)((0.15)(7) + 1)(0.15)^{-2} + (500000)(0.15)^{-1}) \\ &+ (2)(-26578)(0.15)^{-2} + (500000)(0.15)^{-1} = 1499170 \end{aligned} \tag{29}$$

Using Equation (22) and the parameters above the solution to integral three is...

$$I_3 = -\text{Exp}\,\{(-0.15)(7)\}(0.04 - 0.15)^{-1} = 3.1813 \tag{30}$$

Using valuation Equation (23) above, the parameters above and the data in Table 1, the value of our hypothetical company, and the answer to our hypothetical problem, is...

$$\begin{aligned} V &= \theta(1-\alpha)I_1 - \phi I_2 + (\theta\,(1-\alpha) - \phi\,\mu_L)\,R_T\,I_3 \\ &= (0.20)(1-0.30)(9201209) - (0.60)(1499170) + ((0.20)(1-0.30) - (0.60)(0.04))(3197674)(3.1813) \\ &= 1561289 \end{aligned} \tag{31}$$

Appendix

A Integral solution for...

$$\int\limits_u^v \Big(a\,t^2 + b\,t + c\Big)\text{Exp}\Big\{-\kappa\,t\Big\}\delta t = -\text{Exp}\Big\{-\kappa\,t\Big\}\Big(a\Big(\kappa^2t^2 + 2\kappa t + 2\Big)\kappa^{-3} + b\Big(\kappa t + 1\Big)\kappa^{-2} + c\,\kappa^{-1}\Big)\Big[_u^v \tag{32}$$

B Integral solution for...

$$\int\limits_u^v \Big(2\,a\,t + b\Big)\text{Exp}\Big\{-\kappa\,t\Big\}\delta t = -\text{Exp}\Big\{-\kappa\,t\Big\}\Big(2a\Big(\kappa t + 1\Big)\kappa^{-2} + b\,\kappa^{-1}\Big)\Big[_u^v \tag{33}$$

C Integral solution for...

$$\int\limits_u^v \text{Exp}\Big\{\mu_L\Big(t - T\Big) - \kappa\,t\Big\}\,\delta t\,\Big[_u^v = \text{Exp}\Big\{\mu_L\Big(t - T\Big) - \kappa t\Big\}\Big(\mu_L - \kappa\Big)^{-1} \tag{34}$$